MATROID BASE POLYTOPE DECOMPOSITION II : SEQUENCE OF HYPERPLANE SPLITS

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Abstract. This is a continuation of the early paper [3] concerning matroid base polytope decomposition. Here, we will present sufficient conditions on M so its base matroid polytope $P(M)$ has a *sequence* of hyperplane splits. The latter yields to decompositions of $P(M)$ with two or more pieces for infinitely many matroids M. We also present necessary conditions on the Euclidean representation of rank three matroids M for the existences of decompositions of $P(M)$ into 2 or 3 pieces. Finally, we prove that $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

Keywords: Matroid base polytope, polytope decomposition

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1. Introduction

This paper is a continuation of the paper [3] by the two present authors. For general background in matroid theory we refer the reader to [14, 17]. A *matroid* $M = (E, \mathcal{B})$ of *rank* $r = r(M)$ is a finite set $E = \{1, \ldots, n\}$ (called the *ground* set of M) together with a nonempty collection $\mathcal{B} = \mathcal{B}(M)$ of r-subsets of E (called the *bases* of M) satisfying the following *basis exchange axiom*:

if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exists $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

We denote by $\mathcal{I}(M)$ the family of *independent* sets of M (consisting of all subsets of bases of M). For a matroid $M = (E, \mathcal{B})$, let $P(M)$ be the *matroid base polytope* of M defined as the convex hull of the incidence vectors of bases of M, that is,

$$P(M) := \text{conv}\left\{\sum_{i\in B} e_i : B \text{ a base of } M\right\},$$

where e_i denotes the i^{th} standard basis vector in $\mathbb{R}^n$. $P(M)$ is a polytope of dimension at most $n-1$.

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A *matroid base polytope decomposition* of $P(M)$ is a decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where each $P(M_i)$ is also a matroid base polytope for some matroid M_i, and for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is a face of both $P(M_i)$ and $P(M_j)$. It is known that nonempty faces of matroid base polytope are matroid base polytopes [5, Theorem 2]. So, the common face $P(M_i) \cap P(M_j)$ (whose vertices correspond to elements of $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$) must also be a matroid base polytope.

$P(M)$ is said to be *decomposable* if it has a matroid base polytope decomposition with $t \geq 2$ and *indecomposable* otherwise. A decomposition is called *hyperplane split* when $t = 2$.

Matroid base polytope decomposition were introduced by Lafforgue [11, 12] and have appeared in many different contexts (see for instance, [1, 2, 4, 6, 7, 9, 10, 13, 15, 16]). It is thus of interest to investigate whether a given matroid base polytope is decomposable or not.

In [3], we have studied the existence (and nonexistence) of such decompositions. Among other results, we gave sufficient conditions for a matroid M so that $P(M)$ has a hyperplane split (allowing us to construct infinite families with *different* hyperplane splits).

A natural question is the following one: given a matroid base polytope $P(M)$, is it possible to find a sequence of hyperplane splits arising a decomposition of $P(M)$? In other words, is there a hyperplane split of $P(M)$ such that one of the two obtained pieces has a hyperplane split such that, in turn, one of the two new obtained pieces has a hyperplane split, and so on, giving a decomposition of $P(M)$?

In [8, Section 1.3], Kapranov showed that all decompositions of a rank two matroid (appropriately parametrized) can be achieved by a sequence of hyperplane splits. However, this is not always the case, indeed, Billera, Jia and Reiner [2] proved that $P(M)$ can be split into three indecomposable pieces where M is the rank three matroid on $\{1, \ldots, 6\}$ having every triple but $\{1,2,3\}, \{1,4,5\}$ and $\{3,5,6\}$ as bases and that this decomposition cannot be obtained via hyperplane splits. We notice that $P(M)$ also has a decomposition in three pieces, obtained via hyperplane splits, this is illustrated in Example 3.

A difficulty to face when applying hyperplane splits successively is that the intersection $P(M_i) \cap P(M_j)$ must be also a matroid base polytope. For instance, if we consider a sequence of 2 hyperplane splits, say $P(M) = P(M_1) \cup P(M_1')$ and $P(M_1') = P(M_2) \cup P(M_2')$. This sequence would give a decomposition $P(M) = P(M_1) \cup P(M_2) \cup P(M_2')$ if $P(M_1) \cap P(M_2), P(M_1) \cap P(M_2')$ and $P(M_2) \cap P(M_2')$ are matroid base polytopes.

By definition of hyperplane split, $P(M_2) \cap P(M_2')$ will be fine, however the other two intersections might not be matroid base polytopes. Recall that the intersection of two matroids is not necessarily a matroid (for instance, $\mathcal{B}(M_1) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ and $\mathcal{B}(M_2) = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ are matroids while $\mathcal{B}(M_1) \cap \mathcal{B}(M_2) = \{\{1,3\},\{2,3\},\{2,4\}\}$ is not).

In the next section, we give sufficient conditions on M so that $P(M)$ admits a sequence of $t \geq 2$ hyperplane splits. This allows us to give decompositions of $P(M)$ with $t+1$ pieces for infinitely many matroids. In Section 3, we present necessary geometric conditions (on the Euclidean representation) of rank three matroids M for the existences of decompositions of $P(M)$ into 2 or 3 pieces. Finally, in Section 4, we show that the *direct sum* $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

2. Sequence of hyperplane splits

Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $A \subseteq E$. We recall that the independent sets of the *restriction* matroid of M to A, denoted by $M|_A$, are given by $\mathcal{I}(M|_A) = \{I \subseteq A : I \in \mathcal{I}(M)\}$.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a partition of $E = \{1, \dots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \dots, t$. We say that $\bigcup_{i=1}^{t} E_i$ is a *good t-partition* if there exist integers $0 < a_i < r_i$ with the following properties

$(P1)$ $r = \sum_{i=1}^{t} a_i$ and

$(P2)$ (a) For any $1 \leq j \leq t-1$

$$\begin{array}{l} \text{if } X \in \mathcal{I}(M|_{E_1 \cup \dots \cup E_j}) \text{ with } |X| \leq a_1 \\ \text{and } Y \in \mathcal{I}(M|_{E_{j+1} \cup \dots \cup E_t}) \text{ with } |Y| \leq a_2 \\ \text{then } X \cup Y \in \mathcal{I}(M). \end{array}$$

(b) For any pair $1 \leq j < k \leq t-1$

$$\begin{array}{ll} \text{if } X \in \mathcal{I}(M|_{E_1 \cup \dots \cup E_j}) & \text{with } |X| \leq \sum_{i=1}^{j} a_i \\ \text{and } Y \in \mathcal{I}(M|_{E_{j+1} \cup \dots \cup E_k}) & \text{with } |Y| \leq \sum_{i=j+1}^{k} a_i \\ \text{and } Z \in \mathcal{I}(M|_{E_{k+1} \cup \dots \cup E_t}) & \text{with } |Z| \leq \sum_{i=k+1}^{t} a_i \\ \text{then } X \cup Y \cup Z \in \mathcal{I}(M). & \end{array}$$

Notice that 2-partitions (given by $(P2)$ case (a) with $t = 2$) are the *good partitions* defined in [3]. Good partitions were used to give sufficient conditions for the existence of hyperplane splits. The latter was a consequence of the following two results.

Lemma 1. [3, Lemma 1] *Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then,*

$$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\} \text{ and } \mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_2| \leq a_2\}$$

are the collections of bases of matroids.

Theorem 1. [3, Theorem 1] *Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then, $P(M) = P(M_1) \cup P(M_2)$ is a hyperplane split where M_1 and M_2 are the matroids given in Lemma 1.*

We shall use these two results as the initial step in our construction for a sequence of $t \geq 2$ hyperplane splits.

Lemma 2. *Let $t \geq 2$ be an integer and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, i=1,...,t. Let*

$$\mathcal{B}(M_1) = \{B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1\}$$

and, for each $j = 1, \ldots, t$, let

$$\mathcal{B}(M_j) = \left\{B \in \mathcal{B}(M) : \begin{array}{l} |B \cap E_1| \geq a_1, \\ \vdots \\ |B \cap \bigcup_{i=1}^{j-1} E_i| \geq \sum_{i=1}^{j-1} a_i, \\ |B \cap \bigcup_{i=1}^{j} E_i| \leq \sum_{i=1}^{j} a_i \end{array} \right\}.$$

Then, $\mathcal{B}(M_i)$ is the collection of bases of a matroid for each $i = 1, \ldots, t$.

Proof. By Properties $(P1)$ en $(P2)$, we have that

$$\text{if } X \in \mathcal{I}(M|_{E_1}) \text{ with } |X| \leq a_1 \text{ and } Y \in \mathcal{I}(M|_{E_2 \cup \cdots \cup E_t}) \text{ with } |Y| \leq \sum_{i=2}^{t} a_i$$

then $X \cup Y \in \mathcal{I}(M)$. So, by Lemma 1, $\mathcal{B}(M_1)$ is the collection of bases of a matroid. Now, notice that

$$\mathcal{B}(\overline{M_1}) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1\}$$

is also the collection of bases of a matroid on E. We claim that

$$P(\overline{M_1}) = P(M_2) \cup P(\overline{M_2})$$

is a hyperplane split where

$$\mathcal{B}(M_2) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \text{ and } |B \cap (E_1 \cup E_2)| \leq a_1 + a_2\}$$

and

$$\mathcal{B}(\overline{M_2}) = \{B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \text{ and } |B \cap (E_1 \cup E_2)| \geq a_1 + a_2\}.$$

Indeed, since $\mathcal{B}(\overline{M_1})$ is the collection of bases of a matroid on E then, again, by Properties $(P1)$ and $(P2)$ case (a),

$$\text{if } X \in \mathcal{I}(\overline{M}|_{E_1 \cup E_2}) \text{ with } |X| \leq a_1 + a_2 \text{ and } Y \in \mathcal{I}(\overline{M}|_{E_3 \cup \cdots \cup E_t}) \text{ with } |Y| \leq \sum_{i=3}^{t} a_i$$

then $X \cup Y \in \mathcal{I}(\overline{M})$. So, by Lemma 1, $\mathcal{B}(M_2)$ is the collection of bases of a matroid (and thus $\mathcal{B}(\overline{M_2})$ also is). By inductively applying the above argument to $\overline{M}_j$, it can be easily checked that $\mathcal{B}(M_j)$ is the collection of bases of a matroid for all j. □

Theorem 2. *Let $t \geq 2$ be an integer and let $M = (E, \mathcal{B})$ be a matroid of rank r. Let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Then, $P(M)$ has a sequence of t hyperplane splits yielding to decomposition*

$$P(M) = \bigcup_{i=1}^{t} P(M_i)$$

where M_i, $1 \leq i \leq t$ are the matroids given in Lemma 2.

Proof. By Theorem 1 the result holds for $t = 2$. Moreover, by the inductively construction of Lemma 2, we clearly have that $P(M) = \bigcup_{i=1}^{t} P(M_i)$ with $\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$. We only need to show that $\mathcal{B}(M_j) \cap \mathcal{B}(M_k)$ is the collection of bases of a matroid for any $1 \leq j < k \leq t$. For, by definition, we have

$$\mathcal{B}(M_j) \cap \mathcal{B}(M_k) = \{B \in \mathcal{B}(M) : \text{condition } C_h(B) \text{ is satisfied for all } 1 \leq h \leq k\}$$

where

- $C_h(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{h} E_i| \geq \sum_{i=1}^{h} a_i$ with $A \subseteq E$ and $1 \leq h \leq k$, $h \neq j, k$,
- $C_j(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i$, $A \subseteq E$,

  and
- $C_k(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i$, $A \subseteq E$.

We will check the exchange axiom for any $X, Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $X, Y \in \mathcal{B}(M)$ then for any $e \in X \setminus Y$ there exists $f \in Y \setminus X$ such that $X - e + f \in \mathcal{B}(M)$. We shall verify that $X - e + f \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. We have three cases (depending on the conditions $C_i(X - e)$ that are satisfied).

Case 1) There exists $1 \leq l \leq j$ such that $C_l(X - e)$ is not satisfied. Let us take the minimum of such index l. Since $l \leq j \leq k$ then $C_l(X)$ is satisfied, and since $C_l(X - e)$ is not satisfied then

(a) $\left|X \cap \bigcup\limits_{i=1}^{l} E_i\right| = \sum\limits_{i=1}^{l} a_i$,

(b) $e \in \bigcup\limits_{i=1}^{l} E_i$ and

(c) $\underbrace{|\,(X - e) \cap \bigcup\limits_{i=1}^{l} E_i\,|}_{I_1} = \sum\limits_{i=1}^{l} a_i - 1.$

Since $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$ then

$$\underbrace{|\,Y \cap \bigcup_{i=1}^{l} E_i\,|}_{I_2} \geq \sum_{i=1}^{l} a_i$$

Therefore, by using (c), $I_1, I_2 \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l}) \subseteq \mathcal{I}(M)$ with $|I_1| < |I_2|$. So, there exist $f \in I_2 \setminus I_1 \subset Y \setminus X$ with $I_1 \cup f \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l})$. Thus, $f \in \bigcup\limits_{i=1}^{l} E_i$ and

$$|I_1 \cup f \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - 1. \tag{1}$$

Moreover, since X is a base then $|X| = r = \sum\limits_{i=1}^{t} a_i$ and so, by (a), we have

$$\underbrace{|\,(X - e + f) \cap \bigcup_{i=l+1}^{t} E_i\,|}_{I_3} \overset{(b)}{=} |X \cap \bigcup_{i=l+1}^{t} E_i| = \sum_{i=1}^{t} a_i - \sum_{i=1}^{l} a_i = \sum_{i=l+1}^{t} a_i.$$

We also have that $I_3 \in \mathcal{I}(M|_{E_{l+1} \cup \cdots \cup E_t})$, thus, by $(P2)$ case (b),

$$I_1 \cup f \cup I_3 \in \mathcal{I}(M) \text{ with } |I_1 \cup f \cup I_3| = \sum_{i=1}^{l} a_i - 1 + 1 + \sum_{i=l+1}^{t} a_i = r$$

and so $I_1 \cup f \cup I_3 = X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, that $C_h(X - e + f)$ is verified for each $1 \leq h \leq k$.

(i) $\underline{h < l}$: Since l is minimum then $C_l(X-e)$ is not verified so $C_h(X-e)$ is satisfied for each $1 \leq h < l$ and thus $C_h(X-e+f)$ is also satisfied (we just added a new element).

(ii) $\underline{h = l}$: By equation (1) we have that $C_l(X-e+f)$ is satisfied.

(iii) $\underline{h > l}$: Since $e, f \in \bigcup\limits_{i=1}^{l} E_i$ then

$$|X - e + f \cap \bigcup_{i=1}^{h} E_i| = |X \cap \bigcup_{i=1}^{h} E_i|$$

and thus $C_h(X-e+f)$ is satisfied if and only if $C_h(X)$ is satisfied which is the case by hypothesis since $h > l$.

Case 2) $C_{l'}(X-e)$ is satisfied for all $1 \leq l' \leq j$ and there exists $j+1 \leq l \leq k-1$ such that $C_l(X-e)$ is not satisfied. Let us take the minimum of such index l. Since $C_l(X)$ is satisfied and $C_l(X-e)$ is not then

(a) $\left|X \cap \bigcup\limits_{i=1}^{l} E_i\right| = \sum\limits_{i=1}^{l} a_i$,

(b) $e \in \bigcup\limits_{i=j+1}^{l} E_i$ (since $C_j(X-e)$ is satisfied) and

(c) $\underbrace{|\,(X-e) \cap \bigcup\limits_{i=1}^{l} E_i\,|}_{I_1} = \sum\limits_{i=1}^{l} a_i - 1$.

Since $C_j(X-e)$ is satisfied then

$$\begin{aligned}\underbrace{|\,(X-e) \cap \bigcup_{i=j+1}^{l} E_i\,|}_{I_1} &= |(X-e) \cap \bigcup_{i=1}^{l} E_i| - |(X-e) \cap \bigcup_{i=1}^{j} E_i| \\ &\overset{(c)}{=} \sum_{i=1}^{l} a_i - 1 - \sum_{i=1}^{j} a_i = \sum_{i=j+1}^{l} a_i - 1. \end{aligned} \tag{2}$$

Let $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied then

$$\begin{aligned}\underbrace{|\,Y \cap \bigcup_{i=j+1}^{l} E_i\,|}_{I_2} &= |Y \cap \bigcup_{i=1}^{l} E_i| - |Y \cap \bigcup_{i=1}^{j} E_i| \\ &\geq \sum_{i=1}^{l} a_i - \sum_{i=1}^{j} a_i = \sum_{i=j+1}^{l} a_i.\end{aligned}$$

Since $|I_1| < |I_2|$ then there exist $f \in I_2 \setminus I_1$ such that $I_1 + f \in \mathcal{I}(M|_{E_{j+1}\cup\cdots\cup E_l})$. So, $f \in \bigcup\limits_{i=j+1}^{l} E_i$ and, by (b), we have

$$X - e + f \cap \bigcup_{i=1}^{j} E_i = X \cap \bigcup_{i=1}^{j} E_i.$$

Since X is a base then $X - e + f \cap \bigcup\limits_{i=1}^{j} E_i \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ (also notice that $X - e + f \cap \bigcup\limits_{i=l+1}^{t} E_i \in \mathcal{I}(M|_{E_{l+1} \cup \cdots \cup E_t}))$. Moreover, since $X \in \mathcal{B}_j \cap \mathcal{B}_k$ then $C_j(X)$ is satisfied and thus

$$|(X - e + f) \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i \tag{3}$$

and, by equation (2), we have

$$|(X - e + f) \cap \bigcup_{i=j+1}^{l} E_i| = \sum_{i=j+1}^{l} a_i \tag{4}$$

obtaining that

$$|(X - e + f) \cap \bigcup_{i=l+1}^{t} E_i| = r - \sum_{i=1}^{j} a_i - \sum_{i=j+1}^{l} a_i = \sum_{i=l+1}^{t} a_i.$$

Now, by $(P2)$ case (b), we have

$$\left((X - e + f) \cap \bigcup_{i=1}^{j} E_i\right) \cup \left((X - e + f) \cap \bigcup_{i=j+1}^{l} E_i\right) \cup \left((X - e + f) \cap \bigcup_{i=l+1}^{t} E_i\right) = X - e + f \in \mathcal{I}(M)$$

And, since $|X - e + f| = r$ then $X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, that $C_h(X - e + f)$ is verified for each $1 \leq h \leq k$.

(i) $\underline{h < l \text{ with } h \neq j}$: SInce $C_h(X - e)$ is satisfied, then, by the minimality of l we have that $C_h(X - e + f)$ is satisfied.

(ii) $\underline{h = j}$: By equation (3) we have that $C_j(X - e + f)$ is satisfied.

(iii) $\underline{h = l}$: By equations (3) and (4) we have that $C_l(X - e + f)$ is satisfied.

(iv) $\underline{h > l}$: Since $e, f \in \bigcup\limits_{i=j+1}^{l} E_i$ then

$$|X - e + f \cap \bigcup_{i=1}^{h} E_i| = |X \cap \bigcup_{i=1}^{h} E_i|$$

and thus $C_h(X - e + f)$ is satisfied if and only if $C_h(X)$ is satisfied which is the case by hypothesis since $h > l$.

Case 3) $C_i(X-e)$ is satisfied for every $1 \leq i \leq k$.

Subcase a) $|X - e \cap \bigcup\limits_{i=1}^{k} E_i| = \sum\limits_{i=1}^{k} a_i$. We first notice that that $e \in \bigcup\limits_{i=k+1}^{t} E_i$ (otherwise $|X - e \cap \bigcup\limits_{i=1}^{k} E_i| < |X \cap \bigcup\limits_{i=1}^{k} E_i|$ which is impossible since $C_k(X)$ is satisfied). Now,

$$\underbrace{|\,(X-e) \cap \bigcup_{i=k+1}^{t} E_i\,|}_{I_1} = r - 1 - \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{t} a_i - 1. \tag{5}$$

Let $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied then

$$|Y \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i$$

and so

$$\underbrace{|\,Y \cap \bigcup_{i=k+1}^{t} E_i\,|}_{I_2} \geq \sum_{i=k+1}^{t} a_i$$

Since $|I_1| < |I_2|$ then there exist $f \in I_2 \setminus I_1$ such that $I_1 + f \in \mathcal{I}(M|_{E_{k+1}\cup\cdots\cup E_t})$. So, $f \in \bigcup\limits_{i=k+1}^{t} E_i$ and since $e \in \bigcup\limits_{i=k+1}^{t} E_i$ then

$$(X - e + f) \cap \bigcup_{i=1}^{k} E_i = X \cap \bigcup_{i=1}^{k} E_i \in \mathcal{I}(M|_{E_1\cup\cdots\cup E_k})$$

.

Also, since $(X-e+f) \cap \bigcup\limits_{i=k+1}^{t} E_i \in \mathcal{I}(M|_{E_{k+1}\cup\cdots\cup E_t})$ then by $(P2)$ case (b) we have that

$$X - e + f = \left(X - e + f \cap \bigcup_{i=1}^{k} E_i\right) \cup \left(X - e + f \cap \bigcup_{i=k+1}^{t} E_i\right) \in \mathcal{I}(M).$$

Moreover, by using equation (5) and the fact that $f \in \bigcup\limits_{i=k+1}^{t} E_i$ we obtain

$$|X - e + f \cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=k+1}^{t} a_i$$

and since $|X - e \cap \bigcup\limits_{i=1}^{k} E_i| = \sum\limits_{i=1}^{k} a_i$ then

$$|X - e + f \cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i.$$

Therefore,

$$|X - e + f \cap \bigcup_{i=1}^{t} E_i| = |X - e + f \cap \bigcup_{i=1}^{k} E_i| + |X - e + f \cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=1}^{t} a_i = r$$

and so $X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, that $C_h(X - e + f)$ is verified for each $1 \leq h \leq k$. But, since $e, f \in \bigcup\limits_{i=k+1}^{t} E_i$ then $C_h(X - e + f)$ becomes to $C_h(X)$ for all $1 \leq h \leq k$ which is satisfied.

Subcase b) If $|X - e \cap \bigcup\limits_{i=1}^{k} E_i| < \sum\limits_{i=1}^{k} a_i$ then we have that $e \in \bigcup\limits_{i=j+1}^{t} E_i$ (otherwise $|X - e \cap \bigcup\limits_{i=1}^{j} E_i| < |X \cap \bigcup\limits_{i=1}^{j} E_i|$ which is impossible since $C_j(X)$ is satisfied). Now, since $C_j(X - e)$ is satisfied then

$$|(X - e) \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i$$

and thus

$$\underbrace{|\,(X - e) \cap \bigcup_{i=j+1}^{t} E_i\,|}_{I_1} = \sum_{i=j+1}^{t} a_i - 1.$$

Let $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $C_j(Y)$ and $C_l(Y)$ are satisfied then

$$|Y \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i$$

and thus

$$\underbrace{|\,Y \cap \bigcup_{i=j+1}^{t} E_i\,|}_{I_2} = \sum_{i=j+1}^{t} a_i$$

Since $|I_1| < |I_2|$ then there exist $f \in I_2 \setminus I_1$ such that $I_1 + f \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})$. So, $f \in \bigcup\limits_{i=j+1}^{t} E_i$. Since $e \in \bigcup\limits_{i=j+1}^{t} E_i$ then

$$X - e + f \cap \bigcup_{i=1}^{j} E_i = X \cap \bigcup_{i=1}^{j} E_i \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j}) \tag{6}$$

and, by $(P2)$ case (b), we have

$$\left(X - e + f \cap \bigcup_{i=1}^{j} E_i\right) \cup \left(X - e + f \cap \bigcup_{i=j+1}^{t} E_i\right) \in \mathcal{I}(M)$$

Therefore, $X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is, that $C_h(X - e + f)$ is verified for each $1 \leq h \leq k$.

(i) $\underline{h < j}$: Since $C_h(X - e)$ is satisfied then $C_h(X - e + f)$ is also satisfied.

(ii) $\underline{h = j}$: $C_j(X - e + f)$ is satisfied by equation (6).

(iii) $\underline{j + 1 \leq h \leq k - 1}$: Since $C_h(X - e)$ is satisfied then $C_h(X - e + f)$ is also satisfied.

(iii) $\underline{h = k}$: Since $|X - e \cap \bigcup_{i=1}^{k} E_i| < \sum_{i=1}^{k} a_i$ then $|X - e + f \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i$ and thus $C_h(X - e + f)$ is satisfied. □

2.1. **Uniform matroids.** We say that two decompositions $P(M) = \bigcup_{i=1}^{t} P(M_i)$ and $P(M) = \bigcup_{i=1}^{t} P(M'_i)$ are *equivalent* if there is a permutation σ of $\{1, \ldots, t\}$ such that $P(M_i)$ is *combinatorial equivalent* to $P(M'_{\sigma(i)})$ (that is, the corresponding face lattices are isomorphic). They are *different* otherwise.

Corollary 1. *Let $n, r, t \geq 2$ be integers with $n \geq r + t$ and $r \geq t$. Let $p_t(n)$ be the number of different decompositions of the integer n of the form $n = \sum_{i=1}^{t} p_i$ with $p_i \geq 2$ and let $h_t(U_{n,r})$ be the number of decompositions of $P(U_{n,r})$ into t pieces. Then,*

$$h_t(U_{n,r}) \geq p_t(n).$$

Proof. We consider the partition $E = \{1, \ldots, n\} = \bigcup_{i=1}^{t} E_i$ where

$$\begin{aligned} E_1 &= \{1, \ldots, p_1\} \\ E_2 &= \{p_1 + 1, \ldots, p_1 + p_2\} \\ &\vdots \\ E_t &= \{\sum_{i=1}^{t-1} p_i + 1, \ldots, \sum_{i=1}^{t} p_i\}. \end{aligned}$$

We claim that $\bigcup_{i=1}^{t} E_i$ is a good t-partition. For, we first notice that $M|_{E_i}$ is isomorphic to $U_{p_i,\min\{p_i,r\}}$ for each $i = 1, \ldots, t$. Let $r_i = r(M|_{E_i}) = \min\{p_i, r\}$. We now show that

$$\sum_{i=1}^{t} r_i \geq r + t. \tag{7}$$

For, we note that

$$\sum_{i=1}^{t} r_i = \sum_{i=1}^{t} r(M|_{E_i}) = \sum_{i \in T \subseteq \{1,\ldots,t\}} p_i + (t - |T|)r.$$

We have three cases.

1) If $t = |T|$ then $\sum\limits_{i=1}^{t} r_i = \sum\limits_{i=1}^{t} p_i = n \geq r + t$.

2) If $t = |T| + 1$ then $\sum\limits_{i=1}^{t} r_i = \sum\limits_{i=1}^{t-1} p_i + r \geq 2(t-1) + r \geq t + t - 2 + r \geq t + r$.

3) If $t = |T| + k$ with $k \geq 2$ then $\sum\limits_{i=1}^{t} r_i \geq kr \geq 2r \geq r + t$.

So, by equation (7), we can find integers $a'_i \geq 1$ such that $\sum\limits_{i=1}^{t} r_i = r + \sum\limits_{i=1}^{t} a'_i$ Therefore, there are integers $a_i = r(M|_{E_i}) - a'_i$ with $0 < a_i < r(M|_{E_i})$ such that $r = \sum\limits_{i=1}^{t} a_i$. Moreover, if $X \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq \sum\limits_{i=1}^{j} a_i$, $Y \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_k})$ with $|Y| \leq \sum\limits_{i=j+1}^{k} a_i$ and $Z \in \mathcal{I}(M|_{E_{k+1} \cup \cdots \cup E_t}$ with $|Z| \leq \sum\limits_{i=k+1}^{t} a_i$ for $1 \leq j < k \leq t-1$ then $|X \cup Y \cup Z| \leq \sum\limits_{i=1}^{t} a_i = r$ and so $X \cup Y \cup Z$ is always a subset of one of the bases of $U_{n,r}$. Thus, $X \cup Y \cup Z \in \mathcal{I}(U_{n,r})$ and $(P2)$ is also verified. $\square$

Notice that there might be several choices for the values of a_i (each of which arises to good t-partition). However, it is not clear if these give different sequences of t hyperplane splits.

Example 1: Let us consider $U_{8,4}$. We take the partition $E_1 = \{1,2\}$, $E_2 = \{3,4\}$, $E_3 = \{5,6\}$ and $E_4 = \{7,8\}$. We have that $r(M|_{E_i}) = 2$, $i = 1,\ldots,4$. It is easy to check that if we set $a_i = 1$ for each i then $E_1 \cup E_2 \cup E_3 \cup E_4$ is a good 4-partition and thus $P(U_{8,3}) = P(M_1) \cup P(M_2) \cup P(M_3) \cup P(M_4)$ is a decomposition where

$$\begin{aligned}
&\mathcal{B}(M_1) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \leq 1\},\\
&\mathcal{B}(M_2) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \leq 1\},\\
&\mathcal{B}(M_3) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \geq 1,\ |B \cap \{5,6\}| \leq 1\}\\
&\mathcal{B}(M_4) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1,2\}| \geq 1,\ |B \cap \{3,4\}| \geq 1,\ |B \cap \{5,6\}| \geq 1\}.
\end{aligned}$$

2.2. **Relaxations.** Let $M = (E, \mathcal{B})$ be a matroid of rank r and let $X \subset E$ be both a circuit and a hyperplane of M (recall that a *hyperplane* is a *flat*, that is $X = cl(X)$, of rank $r-1$). It is known [14, Proposition 1.5.13] that $\mathcal{B}(M') = \mathcal{B}(M) \cup X$ is the collection of bases of a matroid M' (called, *relaxation* of M).

Corollary 2. *Let $M = (E, \mathcal{B})$ be a matroid and let $E = \bigcup_{i=1}^{t} E_i$ be a good t-partition. Then, $P(M')$ has a sequence of t hyperplane splits where M' is a relaxation of M.*

Proof. It can be checked that the desired sequence of t hyperplane splits of $P(M')$ can be obtained by using the same given good t partition $E = \bigcup_{i=1}^{t} E_i$. □

We notice that the above result is not the only way to give a sequence of hyperplane splits for relaxations. Indeed it is proved in [3] that binary matroids (and thus graphic matroids) do not have hyperplane splits, however we can give a sequence of hyperplane splits for relaxations of graphic matroids as it is shown in Example 3 below.

3. Rank three matroids: geometric point of view

We recall that a matroid of rank three on n elements can be represented geometrically by placing n points on the plane such that if three elements form a circuit, the corresponding points are collinear (in such diagram the lines need not be straight). Then the bases of M are all subsets of points of cardinal 3 which are not collinear in this diagram. Conversely, any diagram of points and lines in the plane in which a pair of lines meet in at most one point represents a unique matroid whose bases are those 3-subsets of points which are not collinear in this diagram.

The combinatorial conditions $(P1)$ and $(P2)$ can be translated into geometric conditions when M is of rank three. The latter is given by the following two corollaries.

Corollary 3. *Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2$ be a partition of the points of the geometric representation of M such that*

1) $r(M|_{E_1}) \geq 2$ and $r(M|_{E_2}) = 3$ and

2) for each line l of M, if $|l \cap E_1| \neq \emptyset$ then $|l \cap E_2| \leq 1$.

Then, $E = E_1 \cup E_2$ is a 2-good partition.

Proof. $(P2)$ case (a) can be easily checked with $a_1 = 1$ and $a_2 = 2$. □

Example 2. Let M be the rank three matroid arising from the configuration of points given in Figure 1. It can be easily checked that $E_1 = \{1, 2\}$ and $E_2 = \{3, 4, 5, 6\}$ verify conditions in Corollary 3. Thus, $E_1 \cup E_2$ is a 2-good partition

Corollary 4. *Let M be a matroid of rank 3 on E and let $E = E_1 \cup E_2 \cup E_3$ be a partition of the points of the geometric representation of M such that*

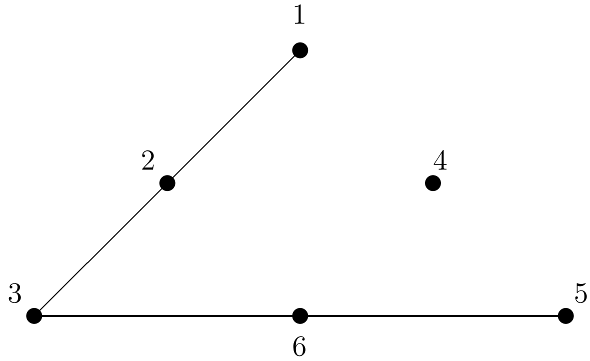

FIGURE 1. Set of points in the plane

1) $r(M|_{E_i}) \geq 2$ for each $i = 1, 2, 3$.
2) For each line l with at least 3 points of M,
a) if $|l \cap E_1| \neq \emptyset$ then $|l \cap (E_2 \cup E_3)| < 2$,
b) if $|l \cap E_3| \neq \emptyset$ then $|l \cap (E_1 \cup E_2)| < 2$.
Then, $E = E_1 \cup E_2 \cup E_3$ is a 3-good partition.

Proof. $(P2)$ case (a) and case (b) can can be easily checked with $a_1 = a_2 = a_3 = 1$. □

Example 3. Let W^3 be the matroid of rank 3 on $E = \{1, \ldots, 6\}$ having as set of bases all 3-subsets of E except the triples $\{1, 2, 3\}$, $\{1, 4, 5\}$ and $\{3, 5, 6\}$. W^3 correspond to the example given by Billera et *al.* [2] that we mentioned at the end of the introduction, see Figure 2. Notice that W^3 is a relaxation of $M(K_4)$ (by relaxing circuit $\{2, 4, 6\}$) and that it is not graphic.

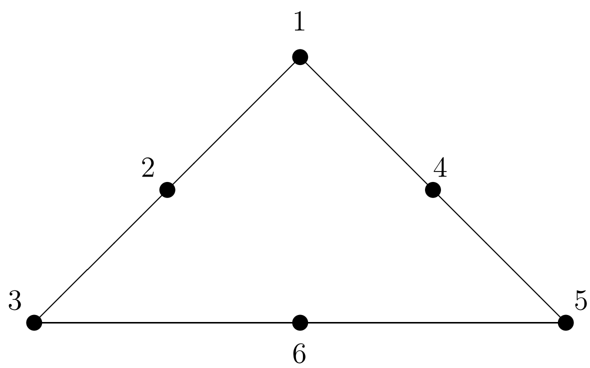

FIGURE 2. Euclidean representation of W^3

It can be checked that $E_1 = \{1, 6\}$ and $E_2 = \{2, 5\}$ and $E_2 = \{1, 4\}$ verify conditions in Corollary 4. Thus, $E_1 \cup E_2 \cup E_3$ is a good 3-partition.

We finally notice that given the 2-good partition $E_1 \cup E_2$ of the matroid M in Exemple 2, we can apply a hyperplane split to the matroid $M|_{E_2}$ induced by the set of points in $E_2 = \{3,4,5,6\}$. Indeed, it can be checked that $E_2^1 = \{3,4\}$ and $E_2^2 = \{5,6\}$ verify conditions in Corollary 3 and thus it is a good 2-partition of $M|_{E_2}$. Moreover, it can be checked that $E_1 = \{1,2\}$, $E_2^1 = \{3,4\}$ and $E_2^2 = \{5,6\}$ verify conditions in Corollary 4. and thus $E_1 \cup E_2 \cup E_3$ is a good 3-partition for M.

4. Direct sum

Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. The *direct sum*, denoted by $M_1 \oplus M_2$, of matroids M_1 and M_2 has as ground set the disjoint union $E(M_1 \oplus M_2) = E(M_1) \cup E(M_2)$ and as set of bases $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 | B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$. Further, the rank of $M_1 \oplus M_2$ is $r_1 + r_2$.

In [3], it is proved the following result.

Theorem 3. [3] *Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ has a hyperplane split if and only if either $P(M_1)$ or $P(M_2)$ has a hyperplane split.*

Our main result in this section is the following.

Theorem 4. *Let $M_1 = (E_1, \mathcal{B})$ and $M_2 = (E_2, \mathcal{B})$ be matroids of rank r_1 and r_2 respectively where $E_1 \cap E_2 = \emptyset$. Then, $P(M_1 \oplus M_2)$ admits a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ admits a sequence of hyperplane splits.*

Proof. Without loss of generality, we suppose that $P(M_1)$ as a sequence of hyperplane splits yielding to the decomposition $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$. For each $i = 1, \ldots, t$, we let

$$L_i = \{X \cup Y : X \in \mathcal{B}(N_i), Y \in \mathcal{B}(M_2)\}$$

Since N_i and M_2 are matroids then L_i is also the matroid given by $N_i \oplus M_2$.

Now for all $1 \leq i, j \leq t$, $i \neq j$ we have

$$L_i \cap L_j = \{X \cup Y : X \in \mathcal{B}(N_i) \cap \mathcal{B}(N_j), Y \in \mathcal{B}(M_2)\}$$

Since $\mathcal{B}(N_i) \cap \mathcal{B}(N_j) = \mathcal{B}(N_i \cap N_j)$ and M_2 are matroids then $L_i \cap L_j$ is also a matroid given by $(N_i \cap N_j) \oplus M_2$. Moreover, $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ alors $\mathcal{B}(M_1) = \bigcup_{i=1}^{t} \mathcal{B}(N_i)$ and thus

$$\begin{aligned}
\bigcup_{i=1}^{t} L_i &= \{X \cup Y : X \in \bigcup_{i=1}^{t} \mathcal{B}(N_i), Y \in \mathcal{B}(M_2)\} \\
&= \{X \cup Y : X \in \mathcal{B}(M_1), Y \in \mathcal{B}(M_2)\} \\
&= \mathcal{B}(M_1 \oplus M_2).
\end{aligned}$$

We now show that this matroid base decomposition induces a t-decomposition of $P(M_1 \oplus M_2)$. Indeed, we claim that $P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i)$ For, we proceed by induction on t. The case $t = 2$ is true since, in the proof of Theorem 3, was showed that $P(M_1 \oplus M_2) = P(L_1) \cup P(L_2)$. We suppose that the result is true for t and let

$$P(M_1) = \bigcup_{i=1}^{t-1} P(N_i) \cup P(N_t^1) \cup P(N_t^2) \tag{8}$$

where N_i, $i = 1, \dots t-1$, N_t^1, N_t^2 are matroids. Moreover, we suppose that throughout the sequence of hyperplane splits of $P(M_1)$ we had $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ and that the last hyperplane split was applied to $P(N_t)$ (obtaining $P(N_t) = P(N_t^1) \cup P(N_t^2)$) and yielding to equation (8).

Now, by the inductive hypothesis, the decomposition $P(M_1) = \bigcup_{i=1}^{t} P(N_i)$ implies the decomposition $P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i)$. But, by the case $t = 2$, $P(N_t) = P(N_t^1) \cup P(N_t^2)$ implying the decomposition $P(N_t \oplus M_2) = P(L_t^1) \cup P(L_t^2)$ where

$L_t^1 = \{X \cup Y : X \in \mathcal{B}(N_t^1), Y \in \mathcal{B}(M_2)\}$ and $L_t^2 = \{X \cup Y : X \in \mathcal{B}(N_t^2), Y \in \mathcal{B}(M_2)\}$

Therefore,

$$P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i) = \bigcup_{i=1}^{t-1} P(L_i) \cup P(L_t^1) \cup P(L_t^2).$$

$\square$

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